From additive to multiplicative thinking – The big challenge of the middle years[1]

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The transition from additive to multiplicative thinking is one of the major barriers to learning mathematics in the middle years. This workshop will explore some of the tasks from a current research project that are being used to identify steps in the development of multiplicative thinking from Years 4 to 8 in a number of Victorian and Tasmanian schools.

Background

A principal aim of teaching and learning mathematics is to help the learner create meaningful mental objects that can be manipulated, considered, and used flexibly and creatively to achieve some purpose. This requires that teachers are knowledgeable of developmental pathways and key learning trajectories, so that all students at all levels have the opportunity to learn the mathematics they need to progress to further study and effective, rewarding citizenship.

It is no longer acceptable that students leave school without the foundation knowledge, skills and dispositions they need to be able to function effectively in modern society. This includes the ability to read, interpret and act upon a much larger range of texts than those encountered by previous generations. An analysis of commonly encountered texts found that approximately 90% were identified as requiring some degree of quantitative and/or spatial reasoning. Of these texts, the mathematical knowledge most commonly required was some understanding of rational number and proportional reasoning, that is, fractions, decimals, percent, ratio and proportion. Multiplicative thinking is a pre-requisite for working with these powerful and necessary ideas. Students cannot be expected to understand and use rational number ideas and representations with any confidence if their understanding of multiplication (and division) is restricted to a 'groups of' model with small whole numbers.

The *Middle Years Numeracy Research Project* (MYNRP), conducted in Victoria from November 1999 to November 2000, used relatively open-ended, 'rich assessment' tasks to measure the numeracy performance of approximately 7000 students in Years 5 to 9. The tasks valued mathematical content knowledge as well as strategic and contextual knowledge and generally allowed all learners to make a start.

For the purposes of the MYNRP, numeracy in the middle years was seen to involve

- core mathematical knowledge, in this case, number sense, measurement and data sense and spatial sense as elaborated in the National Numeracy Benchmarks for Years 5 and 7 (1997);
- the capacity to critically apply what is known in a particular context to achieve a desired purpose; and
- the actual processes and strategies needed to communicate what was done and why.

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[1] In J. Mousley, L Bragg, & C. Campbell, (Eds.) *Mathematics – Celebrating Achievement, Proceedings of the 42nd Conference of the Mathematical Association of Victoria*, Melbourne: MAV

Results from the initial data collection suggest that 22.2% of students overall (31% at Year 5, 18% atYear 6, 25% at Year 7, 19% at Year 8, and 18% at Year 9) were relying on simple 'make-all, count-all' models, skip counting by twos or doubling to solve problems that could be solved more efficiently using multiplication.

Data from the final stage of the project indicates that teachers working in professional teams in a coordinated and purposeful way do make a difference to student numeracy outcomes, particularly where there was concerted focus on 'good' mathematics teaching. That is, the use of problem solving, extended discussion, student explanations, rich assessment and a range of materials, tasks and activities. However, the research also suggests that systems and schools still face a significant challenge in recognising and dealing with the issues involved in teaching and learning for numeracy at this level.

'Hotspots' identified by the research suggest that we need to pay careful attention to the 'big ideas' in mathematics and foster students' capacity to critically reflect on their learning. In particular, it would appear that we need to focus on the development of place-value, multiplicative thinking, rational number ideas, and what is needed to help students progress to the next 'big idea' (see Siemon, Virgona & Corneille, 2001).

What is it?

Multiplicative thinking is characterised by:

- a capacity to work flexibly and efficiently with an extended range of numbers (for example, larger whole numbers, decimals, common fractions, ratio, and per cent),
- an ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion, and
- the means to communicate this effectively in a variety of ways (for example, words, diagrams, symbolic expressions, and written algorithms).

In short, multiplicative thinking is indicated by a capacity to work flexibly with the concepts, strategies and representations of multiplication (and division) as they occur in a wide range of contexts. For example from:

3 bags of sweets, 8 sweets in each bag. How many sweets altogether?

to problems such as the following:

Juli bought a dress in an end-of-season sale for $49.35. The original price was covered by a 30% off sticker but the sign on the rack said, "Now an additional 15% off already reduced prices". How could she work out how much she had saved? What percentage of the original cost did she end up paying?

35 feral cats were found in a 146 hectare nature reserve. 27 feral cats were found in a 103 hectare reserve. Which reserve has the biggest feral cat problem?

The presence/absence of multiplicative thinking can be seen in the solution strategies used to solve:

A muffin recipe requires $\frac{2}{3}$ of a cup of milk. Each recipe makes 12 muffins. How many muffins can be made using 6 cups of milk?

A solution which added $\frac{2}{3}$ repeatedly to find that this can be done nine times using 6 cups of milk and then added 12 nine times is indicative of **additive thinking** (see Figure 1).

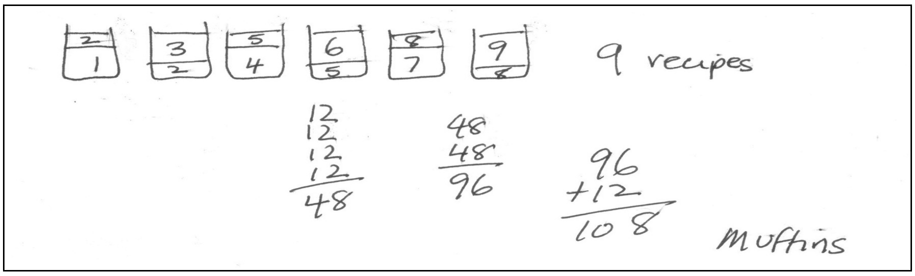


Figure 1 Example of additive solution to Muffin Problem

A solution which determined that 9 recipes could be made on the basis that 3 recipes can be made from 2 cups of milk, then multiplied 9 by 12 to get 108 muffins is indicative of **multiplicative thinking** (see Figure 2).

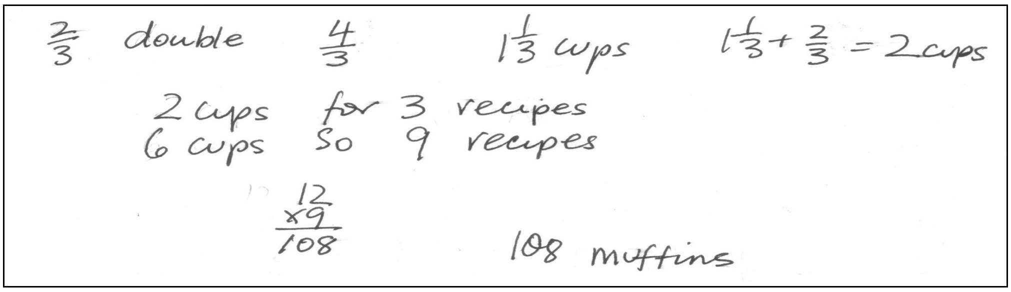
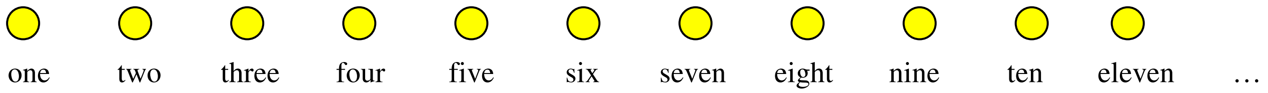


Figure 2 Multiplicative soltion to Muffin Problem

While the every-day experience of most 5 year-olds supports intuitive ideas of getting/having more (addition), losing/having something less (subtraction), fair shares/sharing (division), and doubling, it generally does not support a notion of counting equal groups or repeated addition. In the first two years of schooling, children are introduced to an expanded range of contexts in which addition and subtraction occur, develop efficient mental strategies for addition and subtraction facts to 20, and become acquainted with place-value. As a consequence, by the end of Year 3 most students are able to use a variety of means to solve a range of addition and subtraction problems. If there are problems with addition and subtraction it may mean that students do not *trust the count* (Willis, 2002), that is, they do not have flexible, mental models for the numbers 0 to 10 that enable them to think about these numbers in many ways, for example, 7 is 1 more than 6, 1 less than 8, 3 less than 10, 3 and 4, 2 more than 5 and so on. Referred to as part-part-whole knowledge this is also a critical element in the development of multiplicative thinking.

In contrast to the relative short time needed to develop additive thinking, the introduction and exploration of ideas to support multiplication may take many years and according to some researchers, may not be fully understood by students until they are well into their teen years (Vergnaud, 1988; Clarke & Kamii, 1996; Sullivan et al, 2001).

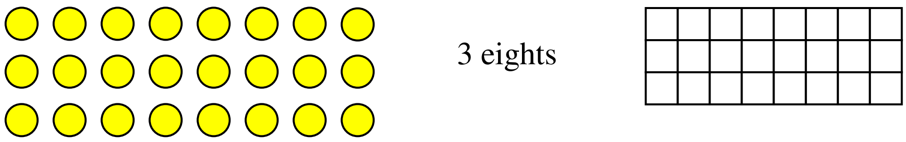
It is generally agreed that the initial idea that needs to be developed is the *groups of* idea. There are a number of ways that this can be done but two of the most useful ways appear to be counting large collections efficiently (for example, by twos, fives or tens and organizing the count) and systematically sharing collections (for example, exploring how many ways 24 counters can be shared equally). The language and recording associated with this idea is also important. Talking about “groups of” or “lots of” can get in the way of understanding what is going on, which is actually a count of a count. This explains to some extent why this idea can be so diffcult for some children who are expected to move from a one-to-one counts as in:



to counting their counting, or a one-to-many count, as in the following:

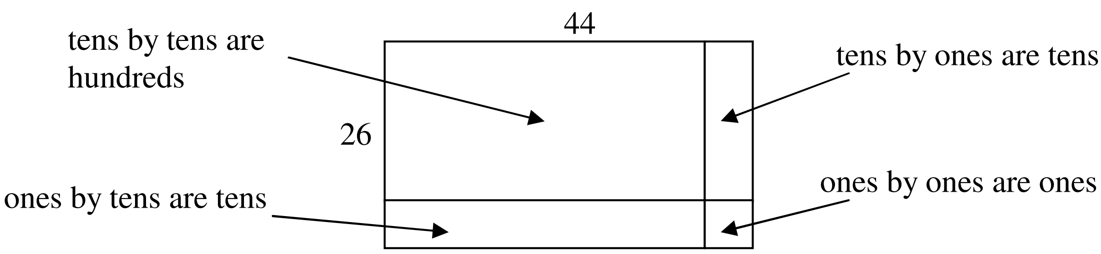
This difficulty is exacerbated by the tendency, particularly in Victoria, to focus on quotition division, that is, counting the number of groups of a known size in a collection (‘goes into’ or *guzinta* idea), at the expense of partition division (sharing) which focuses attention on the number in each of a known number of shares, despite the fact that this is recommended for all formal division from Level 3 on in CSFII.

By distinguishing between the number of groups (4) and what is in the group (threes), students’ attention is drawn directly to the process of counting and what is being counted. Making and naming equal groups is an important experience for Year 2 and 3 students initially which can and should be explored in a variety of ways. One of the advantages of sharing suggested above is that it leads to the realization that a collection may be partitioned in more than one way, in this case, that 24 is 2 twelves, 3 eights, 4 sixes, 6 fours, and 12 twos, each of which can be represented more efficiently by an *array* or a *region*, for example,



A major advantage of arrays is that they can be rotated to show, in this case, that 3 eights is the same as 8 threes. But the real reason that these more efficient representations need to be explored and manipulated is that they support a shift in thinking from counting equal groups to the idea of an equal number of groups of a different size, for example, the traditional ‘3 times table’ counts threes: 1 x **3**, 2 x **3**, 3 x **3**, 4 x **3**, ... and so on. Whereas the array supports the ideas of **3** ones, **3** twos, **3** threes, **3** fours, **3** fives, ... **3** amythings, and it is this idea that is needed to support more efficient mental strategies for the multiplication facts, for example, for 3 eights, double the group (16) and add one more group (24).

While the *array* and *region* ideas embody the *groups of* idea, their real strength lies in the fact that they support the shift referred to above, provide a basis for understanding fraction diagrams, and lead to the *area* idea which is needed to accommodate larger whole numbers and rational numbers, for example, 26 x 44,



which could equally well represent 2.6 by 4.4 where ones by ones are ones, tenths by ones and ones by tenths are tenths, and tenths by tenths are hundredths. This is an important idea which demonstrates how multiplication distributes over addition.

The *area* idea, in turn, generalizes to the *factor-factor-product* idea which is needed to support multiple factor sitations such as 24 = 2 x 2 x 2 x 3, exponentiation as in 4 x 4 x 4, and algebraic factorization as in

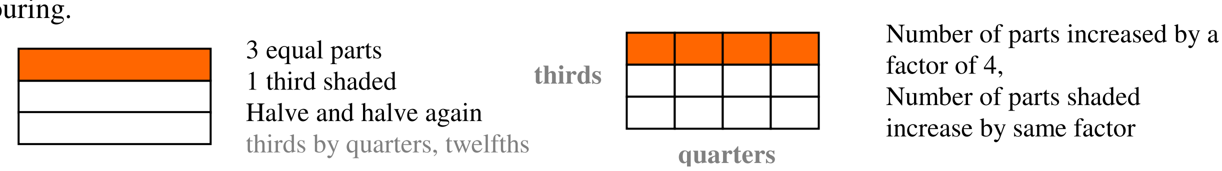
$$(2x + 1)(\ ?\ ) = 2x^2 + 7x + 3$$

Of course, these are not the only ideas for multiplication. Another idea which is found in rational number and Chance and Data contexts is the *for each* idea (more formerly known as the Cartesian product). This arises in situations such as the following.

> I have 4 tops, 3 skirts and 2 pairs of shoes all of which 'go together'. How many different combinations are possible?

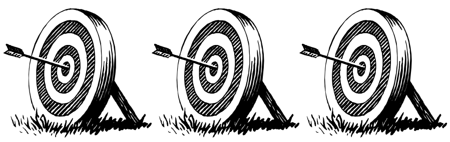
But it also applies in rate or proportion problems and is evident in the structure of the place-value system, where for example, we need to think about the fact that for each ten, there are 10 ones, for each one there are 10 tenths, and for each tenth there are 10 hundredths and so on.

This analysis helps explain why fraction diagrams and fraction renaming (equivalent fractions) are so difficult for so many students in the middle years. If you already have a deep understanding of fractions in terms of partitioning and the ideas of *region* and *for each* for multiplication, then it is a relatively simple jump to interpreting and using fraction diagrams (see below) but if you do not have access to these ideas, an activity such as, "shade to show 2/5" (given a fraction diagram), amounts to little more than counting to 2 and colouring.



Investigating the development of multiplicative thinking

The *Scaffolding Numeracy in the Middle Years* (2003-2006) project[1] was primarily established to identify the key points in the development of multiplicative thinking and rational number beyond the early years. Three school clusters are involved in the research, two in Victoria and one in Tasmania. Each cluster comprises a secondary school and at least three primary schools. Two teachers from each Year level (4 to 8) from each school are participating in the study. A range of tasks were designed or sourced to assess various aspects of multiplicative thinking. An example of one of these tasks is given below.



ADVENTURE CAMP ...

Camp Reefton offers 4 activities. Everyone has a go at each activity early in the week. On Thursday afternoon students can choose the activity that they would like to do again.

The table shows how many students chose each activity at the Year 5 camp and how many chose each activity at the Year 7 camp a week later.

	Rock Wall	Canoeing	Archery	Ropes Course
Year 5	15	18	24	18
Year 7	19	21	38	22

Camp Reefton Thursday Activities

1. What can you say about the choices of Year 5 and Year 7 students?
2. The Camp Director said that canoeing was more popular with the Year 5 students than the Year 7 students. Do you agree with the Director's statement? Use as much mathematics as you can to support your answer.

Before attempting the tasks, teachers worked through two examples to illustrate what was meant by an explanation and the instruction "to use as much mathematics as you can". Table 1 shows the scoring rubric that was used by research school teachers to evaluate student performance on this task.

Table 1

TASK:	RESPONSE:	SCORE
a.	No response or incorrect statement	0
	One or two relatively simple observations based on numbers alone, eg, "Archery was the most popular activity for both Year 5 and Year 7 students", "More Year 7 students liked the rock wall than Year 5 students"	1
	At least one observation which recognises the difference in total numbers, eg, "Although more Year 7s actually chose the ropes course than Year 5, there were less Year 5 students, so it is hard to say".	2
b.	No response	0
	Incorrect (No), argument based on numbers alone, eg, "There were 21 Year 7s and only 18 Year 5s".	1
	Correct (Yes), but little/no working or explanation to support conclusion	2
	Correct (Yes), working and/or explanation indicates that numbers need to be considered in relation to respective totals, eg, "18 out of 75 is more than 21 out of 100", but no formal use of fractions or percent or further argument to justify conclusion	3
	Correct (Yes), working and/or explanation uses comparable fractions or percents to justify conclusion, eg, "For Year 7 it is 21%. For Year 5s, it is 24% because 18/75 = 6/25 = 24/100 = 24%"	4

This proved a useful task in discriminating between additive and multiplicative thinking. For example, students who relied on the relative magnitude of the numbers alone were probably not aware of the relevance of proportion and were working additively (thinking based on the difference between 18 and 21). Those who attempted to relate the numbers to the total number of fifth graders and seventh graders respectively were more likely to be working multiplicatively as they could sense the relevance of proportion in this situation.

Another task, *Missing Numbers*, required students to locate a given set of numbers as accurately as they could on a 0 to 2 number line. The numbers were 1.5, $^3/_4$, 0.2, and $^5/_3$. The second part of this task asked students to provide a detailed justification or explanation for each placement. The trial data provided evidence as to the tolerance limits that discriminated between additive strategies such as count equal parts from the left to the right as opposed to multiplicative strategies that based the placement of the numbers on halving or other partitioning strategies.

Conclusion

In working with the research schools over the last year, it has become very obvious that the move from additive to multiplicative thinking is not trivial. For many students, forced to work with multiplication before they have an adequate understanding of initial ideas such as trusting the count and access to efficient strategies for addition and subtraction, the 'tables' become an object of dread that serve to undermine all their subsequent school mathematics experience. The consequences of not moving beyond a *groups of* idea for multiplication and division (as in guzinta) almost guarantees subsequent failure in relation to developing a deep understanding of fractions, decimals, per cent, ratio and algebra. This is a preventable disease for which we should be 'alert not alarmed'. One way that we can move forward is to slow down and give students time to appreciate the complexities involved. This does not mean 'dumb down' but 'think through'. Think through problem situations, what do they mean, how can they be represented, how can we use what we know, and which strategies are better and why. As teachers we need a deeper understanding of what makes multiplication difficult and how we can scaffold more appropriate strategies. It is hoped that in addition to the rich tasks, the SNMY project will also shed some further light on what is involved in moving from additive to multiplicative thinking.

Endnote:

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